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A Boundary-Layer Problem Associated with Magnetogasdynamic Channel Flow

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In connection with the motion of electrically conducting fluids through electric and magnetic fields, it is useful to consider flow in a channel as a possible means of energy extraction or addition. The simplest form of analysis, based on the familiar one-dimensional approximation, excludes the very important influences of viscous stresses and heat flux. Although one may account for surface friction and heat transfer in an approximate manner as Dahlberg[1] has discussed, there is no possibility for computing the detailed effects. One way of (partially) correcting this defect is based on the idea that, for some distance downstream of the entrance, the immediate effects of the walls may be confined to thin boundary regions. The central portion of the flow is regarded as a one-dimensional problem, the solution of which provides the "freestream" conditions set in the boundary-layer problem.

Although there have been a number of discussions of boundary layers on plates when the fluid is electrically conducting, there seems to have been much less work on problems arising in channel flow. The closest to the present discussion is that by Kerrebrock[2] and Hale,[3] who treat special cases; the latter incorporates Hall currents, which are ignored here. Moffat[4] has discussed the problem and used an integral method to analyze the boundary layers on the side (insulating) walls but with the pressure constant.

The results outlined here are restricted to the necessary conditions for the existence of a class of similarity solutions to a particular set of boundary-layer equations for compressible flow. The formulation is based on certain assumptions that have been discussed in detail, for example, in Refs. 2 and 5. It is necessary only to state the conventions adopted; the electric field E, magnetic induction B, and "freestream" flow speed u are positive in the negative z, positive y, and positive x directions, respectively. The current density j is then positive in the positive z direction for a generator. It is supposed that E and B are established by means external to the flow so that the local current density in a generator is, for a scalar electrical conductivity,

$$j = \sigma(uB - E) \tag{1}$$

If the "magnetic Reynolds number" is small, then the magnetic field associated with the flow of currents in the gas is small, and it is consistent to assume B to be caused essentially by external means only. The corresponding assumption that the electric field strength E also is due to sources outside the flow implies that the gas should be electrically neutral at all points. This situation often prevails because of the large forces that arise if there is significant charge separation. However, in the present problem, somewhat closer examination is necessary. Consider the boundary layers on the electrodes of a generator; away from the regions near the side (insulating) walls, the current must be uniform in the direction normal to the surface since no current flow is permitted, within the approximations adopted here, in the axial directions. Thus, since u and σ vary through the boundary layer, Eq. (1) can be satisfied only if E varies as well. This means that there is space charge within the boundary layer, and the drop in potential across the boundary layer is different from that in an equal distance in the freestream. Equation (1), applied within the boundary layer, is really an equation determining E. The contribution of net charge density, $\epsilon_0\nabla\cdot\mathbf{E}$, to the current flow is a small correction that may be neglected.

On the other hand, the electric field can be uniform within the boundary layers on the side walls except in the regions near the electrodes. One can suppose that the tendency to charge neutrality does prevail and that Eq. (1) is an equation for j in the viscous region. Clearly, at all points for which $uB < E$ ($u \to 0$ near the surface, of course), j is locally negative, and current flows in the direction opposite to that of the current outside the boundary layer. Hence, there exists the possibility for closed-current loops within the channel. The flow in the corners, where the boundary layers on the electrodes and the side walls merge, constitutes a very much more difficult problem that has not been investigated further. It may be a practically important question because of the concomitant power losses.

The situation in an accelerator is different, for $uB < E$ everywhere; the current flows in the same direction at all points. However, since $|j| = \sigma|E - uB|$ and both u and $1/\sigma$ decrease in the region near the side walls, $|j|$ must increase. The flow adjacent to these walls, therefore, offers a "short circuit" path relative to the flow in the central region of the channel. These remarks indicate that the solutions to the boundary-layer equations for the flow over the side walls and electrodes should differ in a qualitative, as well as quantitative, respect, and indeed, one such distinction appears already in the similarity solutions enumerated below.

It is assumed that the fluid behaves as a perfect gas with constant Prandtl number and specific heats. The equations of conservation for the flow outside the boundary layers are those appropriate to a one-dimensional flow; the corresponding boundary-layer equations can be deduced from the Navier-Stokes equations written for an electrically conducting fluid and may be found, for example, in Ref. 3. Following the usual approach, one seeks conditions under which the boundary-layer partial differential equations may be reduced to nonlinear, ordinary differential equations; all

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Table 1a Summary of channel flow solutions for boundary-layer similar flows on electrodes (E_e constant)

	Energy equation	Momentum equation
Constant enthalpy	$n = -E_e j_r \dfrac{A_r x_r}{u_r^2 \rho_e u_e A}$	$\dfrac{u_e}{u_3} = 1 - \left[1 + \dfrac{\alpha(1-3n)}{(5n-1)}\right] \dfrac{(5n-1)}{n(1+\alpha)} \dfrac{1}{\gamma M_e^2}$
Constant velocity	$c = -E_e j_r (1+\alpha) \dfrac{A_r x_r}{u_r^2 \rho_e u_e A}$	$\dfrac{u_e}{u_3} = \left(\dfrac{1-\alpha}{c}\right) \dfrac{\gamma - 1}{\gamma}$
Constant Mach no.	$\lambda = -E_e j_r \dfrac{A_r x_r (1+\alpha)}{2h_r \rho_e u_e A \{1 + [(\gamma-1)/2] M_e^2\}}$	$\dfrac{u_e}{u_3} = \dfrac{M_e^2 + [(1-\lambda-\alpha)/\lambda\gamma]}{M_e^2 + [2/(\gamma-1)]}$

Table 1b Exponents β in the distribution $\psi = \psi_r(x/x_r)^\beta$

ψ	Constant enthalpy	Constant velocity	Constant Mach no.
u_e	n	$\dfrac{c}{1+\alpha}$	$\dfrac{\lambda}{1+\alpha}$
M_e	n	$\dfrac{c}{2(1+\alpha)}$	$\dfrac{2\lambda}{1+\alpha}$
p_e	$\dfrac{1-5n-\alpha(1+n)}{1+\alpha}$	$\dfrac{1-\alpha}{1+\alpha}$	$\dfrac{1-\lambda-\alpha}{1+\alpha}$
ρ_e	$\dfrac{1-5n-\alpha(1+n)}{1+\alpha}$	$\dfrac{1-\alpha-c}{1+\alpha}$	$\dfrac{1-3\lambda-\alpha}{1+\alpha}$
A	$\dfrac{4n-1-\alpha(1+2n)}{1+\alpha}$	$\dfrac{c-1+\alpha}{1+\alpha}$	$\dfrac{2\lambda-1+\alpha}{1+\alpha}$
j	$\dfrac{-2[n+\alpha(1-n)]}{1+\alpha}$	$\dfrac{-2\alpha}{1+\alpha}$	$\dfrac{-2\alpha}{1+\alpha}$
σ_e	$\dfrac{-2\alpha(1-2n)}{1+\alpha}$	$\dfrac{-2\alpha}{1+\alpha}$	$\dfrac{-2\alpha}{1+\alpha}$

dependent variables are then functions of a single variable, and the problem of solution is posed in simplest form. The transformation used here is the familiar stretching of the coordinate normal to the surface, $z \rightarrow \eta$, with $d\eta = (\rho/\rho_e)t(x)dz$, $t(x)$ being a function of x to be determined. Then the class of boundary-layer solutions found are those for which the stream function has the separated form, $\zeta(x)f(\eta)$, with $\zeta(x)$ another function of x to be determined. Simultaneously, the properties of the inviscid flow must also be taken as functions of x consistent with the one-dimensional flow equations. The various functions of x are determined by the requirements of similarity and satisfaction of the equations for the inviscid flow.

One finds eventually that, for the boundary layers on the electrodes, there are three subclasses of the similarity solutions corresponding to constant enthalpy, velocity, and Mach number. The associated channel flow solutions are summarized in Tables 1a and 1b, in which subscript e denotes quantities in the one-dimensional flow and subscript r denotes reference values. For the case quoted, the electric field has been assumed uniform, and the coefficient of viscosity is proportional to the absolute temperature; these restrictions may be relaxed so long as E_e and μ_e vary as powers of x.† One does not have complete freedom in choosing the constants λ, c, n, and α, essentially because of the requirement that $u_e/u_3 > 1$ in a generator and $u_e/u_3 < 1$ is an accelerator;[6] $u_3 = E/B$ is generally a function of x. A summary of the permissible ranges appears in Table 2.

Calculations appropriate to the boundary layers on the side walls can be carried out in a similar manner, but account must be taken that the current density is variable through the boundary layer. It turns out that this further requirement is satisfied only by the similarity solutions for which the "freestream" Mach number is constant. Numerical solutions for this case may, therefore, be the most interesting of those in Table 1.

One might consider the possibility of seeking similar solutions for very large values of M_e by using the stagnation enthalpy rather than the static enthalpy as the dependent variable in the energy equation. It is quite easy to show, however, that there are no such solutions for $M_e \rightarrow \infty$. The difficulty encountered by Kerrebrock[2] arises in all three classes of solutions. No attempt has been made to treat segmented electrodes, which is consistent with the neglect of Hall currents. Similarity solutions cannot be obtained if

Table 2 Summary of permissible ranges of α, n, c, λ, for boundary-layer flows on electrodes (E_e constant)

	Generator ($u_e/u_3 > 1$)		Accelerator ($u_e/u_3 < 1$)	
Constant enthalpy	$\alpha < -1$:	$n < 0$	$\alpha < -1$:	$0 < n < \dfrac{1-\alpha}{5-3\alpha}$
$c = 0$	$1 < \alpha < \frac{5}{3}$:	$\dfrac{\alpha-1}{3\alpha-5} < n < 0$	$-1 < \alpha \leqslant +1$:	$n > \dfrac{1-\alpha}{5-3\alpha}$
$\alpha \neq -1$	$\alpha > \frac{5}{3}$:	$n < 0$	$1 < \alpha \leqslant \frac{5}{3}$:	$n > 0$
			$\alpha \geqslant \frac{5}{3}$:	$n < \dfrac{\alpha-1}{3\alpha-5}$
Constant velocity	$\alpha < -1$:	$0 < c < \dfrac{\gamma-1}{\gamma}(1-\alpha)$		
$\lambda = 0$, $\alpha \neq \pm 1$	$\alpha > +1$:	$-\dfrac{\gamma-1}{\gamma}(\alpha-1) < c < 0$	$-1 < \alpha < +1$:	$c > \dfrac{\gamma-1}{\gamma}(1-\alpha)$
Constant Mach number	$\alpha < -1$:	$0 < \lambda < \dfrac{\gamma-1}{3\gamma-1}(1-\alpha)$	$\alpha < -1$:	$\lambda < 0$
			$-1 < \alpha +1$:	$\lambda > \dfrac{\gamma-1}{3\gamma-1}(1-\alpha)$
$\alpha \neq -1$	$\alpha > +1$:	$\lambda < -\dfrac{\gamma-1}{3\gamma-1}(\alpha-1)$	$\alpha \geqslant 1$:	$\lambda > 0$

† There is also a set of solutions for which all quantities in the inviscid region vary exponentially.[6]

E and j vary discontinuously, but it might be possible to use numerical solutions as a basis for an integral method, as difficult boundary conditions have been handled in other problems.

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Procedure for the Determination of Impact Probabilities

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The various steps required to determine the distribution of hits around a two-dimensional or three-dimensional target caused by random normal errors in the fire-control parameters are discussed. A method is presented to express the distribution as a function of miss distance rather than of the three space variables.

Nomenclature

a,b,c	= distribution parameters
d_{ij}	= elements of D
D	= n by m matrix of partial derivatives $\partial f_i/\partial s_j$
D'	= transposed D matrix
f_i	= functions of fire-control parameters
I	= integral
m	= number of fire-control parameters
n	= number of dimensions, 2 or 3
p	= probability density function
P	= a transformation matrix that yields principal axes
P'	= transposed P matrix
r	= miss distance
s_i	= fire-control parameters
T	= $\mu_2{}^2/\mu_4$
v_{ij}	= elements of V_s
V_s	= m by m variance-covariance matrix of fire-control parameters
V_x	= n by n variance-covariance matrix of impact coordinates
$V_x{}'$	= variance-covariance matrix along principal axes
x_i	= ith impact coordinate
δs_j	= error in jth fire-control parameter
δx_i	= ith component of miss distance
$\delta x_i{}' \equiv \delta_i$	= miss distances along principal axis
$\langle \Delta X\rangle$	= n by 1 matrix containing δx_i
$\langle \Delta S\rangle$	= m by 1 matrix containing δs_j
$\mu_k(r)$	= kth moment of r about origin
$\sigma_{s_j}{}^2$	= variance of jth fire-control parameter
$\sigma_{x_i'}{}^2 \equiv \sigma_i{}^2$	= the eigenvalues of V_x = the diagonal elements of $V_x{}'$ = the variances along principal axes

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Introduction

THE impact coordinates of a missile as a function of fire-control parameters may be predicted by equations of the following form:

$$x_i = f_i(s_1,s_2, \ldots, s_m) \qquad 1 \le i \le n \quad (1)$$

where s_j represents any number of parameters, such as the initial coordinates, velocities, time, etc. Depending on the dimensionality of the target, n may be two or three.

A variation in the impact coordinates δx_i caused by small variations in the parameters δs_j may be expressed by the following equation:

$$\langle \Delta X\rangle = D\langle \Delta S\rangle \quad (2)$$

where D is an n by m matrix, the elements of which are defined by

$$d_{ij} = \partial f_i/\partial s_j \quad (3)$$

Random variations in the parameters are usually assumed to be normally and independently distributed with mean 0 and standard deviation σ_{s_j}. Associated with these random variations is a "variance-covariance" matrix[1] v_s which is of dimensions m by m and has elements defined by

$$\begin{aligned} v_{ij} &= 0 \qquad & i \ne j \\ v_{ij} &= \sigma_{s_j}{}^2 \qquad & i = j \end{aligned} \quad (4)$$

The variance-covariance matrix for the impact coordinates is then found by a simple matrix product

$$V_x = DV_sD' \quad (5)$$

where V_x is a symmetrical n by n matrix.

V_x is the matrix of second moments of the probability density function in the reference frame of the impact coordinates, and in general, off-diagonal terms will appear. It is well known in both statistics and mechanics, where such matrices appear, that there exists a reference frame where the off-diagonal terms will not appear. The transformation equation takes the same form as Eq. 5.[2]

$$V_x = PV_{x'}P' \quad (6)$$

where $V_{x'}$ is a diagonal matrix and P rotates the original axes into principal axes.

The diagonal elements of $V_{x'}$ are the eigenvalues of V_x and may be found by various techniques such as are described in Todd.[3] It is not necessary to find the P matrix.

The elements $\sigma_{x_1'}{}^2$, $\sigma_{x_2'}{}^2$, and $\sigma_{x_3'}{}^2$ (from $V_{x'}$) are the variances along the principal axes and completely describe the probability distribution of hits around the target under the forementioned assumptions.

A simplification in notation is now in order. Let $\sigma_{x_i'}{}^2 \equiv \sigma_i{}^2$ and let the ith component of miss distance along the principal axes be denoted δ_i instead of $\delta x_i{}'$.

The distribution of hits around the target may now be written for the three-dimensional situation:

$$p(\delta_1,\delta_2,\delta_3) = a \exp\{-\tfrac{1}{2}[(\delta_1/\sigma_1)^2 + (\delta_2/\sigma_2)^2 + (\delta_3/\sigma_3)^2]\} \quad (7)$$

where a is the normalizing factor that gives a unit volume under the curve.

Analytically, the problem is completely solved but not in a practical form on which to base human judgment, for in practice one is interested only in how far the target is missed and will have no information as to the direction of the miss. So this information must be deleted from the probability density function above.

The direct approach is to convert to polar or spherical coordinates and integrate out the angular dependence. The integration is possible analytically only in the special situations where all the variances are equal. The distributions ob-